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Optimal Estimates for the Linear Interpolation Error on Simplices

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This paper presents a comprehensive collection of various error estimates for linear interpolation on simplices. The survey covers many relevant aspects such as geometric facts about simplices, functions of several smoothness classes and arbitrary dimension, and different forms of derivatives. Most estimates involve constants that are best possible. Extremal functions and simplices are provided for all sharp inequalities.

Key Words: linear interpolation; interpolation error; simplex.

1. INTRODUCTION

The aim of this paper is to develop optimal estimates for the error in linear interpolation at the vertices of a simplex. Thereby, the intention is to obtain sharp bounds that are independent of the simplex geometry and shape. Only the diameter and two ball radii are used to describe the size of a simplex. We consider the general case in with multivariate, vector-valued functions of arbitrary dimension and degree of smoothness are allowed. The estimates are based only on measures of smoothness of a function and hence do not exploit special function features.

The results can be easily extended to linear vertex splines since the approximation method considered is entirely local. Vertex splines are used in many applications including finite element methods, surface modelling, and visualization. The simplest way to generalize univariate splines with respect to the dimension is to consider products of univariate splines which are called tensor splines. Following this concept, the decomposition of the domain is restricted to multi-dimensional intervals. In contrast, vertex splines based on simplices are more flexible, since they can be used with any domain that can be triangulated. This advantage shows the importance of multivariate vertex splines in the approximation and modelling of surfaces and volumes (see [8] or [9]). For an introduction to multivariate

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splines we refer to [7] and [13]. The dimension and existence of bases for multivariate spline spaces are discussed in [2] and [3]. In view of this strong background the crucial role of good error estimates is obvious.

2. LINEAR INTERPOLATION ON A SIMPLEX

An *m-simplex* S in $\mathbb{R}^n$ is the convex hull of $m+1$ affinely independent vertices $v_0, ..., v_m \in \mathbb{R}^n$. The convex hull of a subset of $v_0, ..., v_m$ with p elements is called a *p-face* of S. One-dimensional faces are *edges*, and faces of dimension $m-1$ are *facets*. An m-simplex is *regular* if all edges are of equal length. The length of the longest edge of S is called the *edge size* or *diameter* h.

For a vector-valued function $f: S \to \mathbb{R}^\ell$,

$$Lf(x) := \sum_{i=0}^{m} \lambda_i(x)\, f(v_i) \tag{1}$$

is the unique linear polynomial that interpolates f at the vertices of S. Here and below, the functions λ_i provide the barycentric coordinates with respect to the vertices of S, i.e., the λ_i are the unique (scalar) linear polynomials on S for which

$$x = \sum_{i=0}^{m} \lambda_i(x)\, v_i, \qquad \sum_{i=0}^{m} \lambda_i(x) = 1, \quad x \in S. \tag{2}$$

Let

$$\|h_1\|_\infty := \max_{x \in S} |h_1(x)|, \qquad \|h_2\|_{2,\infty} := \max_{x \in S} \|h_2(x)\|_2$$

denote the ∞-norm of a scalar function h_1 and a vector-valued or matrix-valued function h_2 on S. We will consider the interpolation error of the form

$$E_\infty := E_\infty(f, S) := \|f - Lf\|_{2,\infty}.$$

It is well known that this interpolation error is below two times the distance between f and the space of all linear polynomials on S (see [4, p. 40]).

3. SIMPLEX BALLS

We commence the derivation of best error bounds for E_∞ with an investigation of balls around simplices. Let S be an m-simplex with vertices

$v_0, ..., v_m$ in $\mathbb{R}^n$, and $B = B(c, r)$ an m-dimensional ball with center c and radius r in $\mathbb{R}^n$. The ball B *circumscribes* S if all vertices $v_0, ..., v_m$ lie on the boundary of B. B is called a *simplex ball* of S if $S \subseteq B$ and if B has the smallest radius among all balls containing S. If B circumscribes a face of S, then B is called a *face ball*. A face ball is *enfolding* if it contains the whole simplex S, and is *valid* if its center c lies within the face. The first proposition states existence and uniqueness of circumscribed balls and simplex balls.

PROPOSITION 3.1. *Let S be an m-simplex in $\mathbb{R}^n$. Then, the following is true*:

(i) *S has one and only one circumscribed ball in $\mathbb{R}^n$, denoted by B_C.*

(ii) *S has one and only one simplex ball in $\mathbb{R}^n$, denoted by B_S.*

Proof. (i). The case $m = n$ is a well-known fact which results, e.g., from the ball equation (see [1, p. 249]). The case $m < n$ can be proved through a projection of $\mathbb{R}^n$ onto the affine hull of S.

(ii) Since S is compact, existence and uniqueness of the simplex ball are immediately obvious.

Figure 1 shows the circumscribed ball B_C, the simplex ball B_S, and two face balls B_1 and B_2 of a 2-dimensional simplex. In this example, c_S lies on the longest edge of the triangle. Besides the definition, a simplex ball

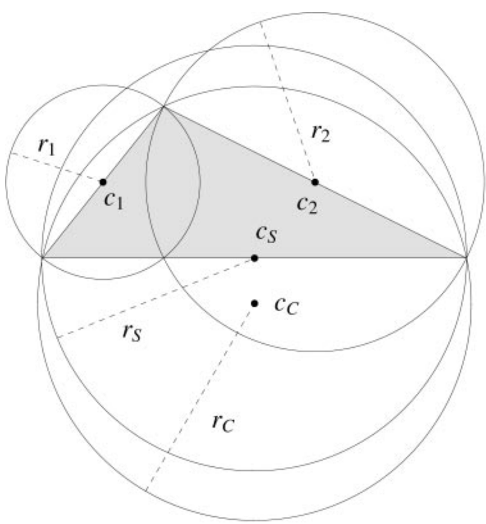

FIG. 1. Circumscribed ball $B_C(c_C, r_C)$, simplex ball $B_S(c_S, r_S)$, and face balls $B_1(c_1, r_1)$ and $B_2(c_2, r_2)$ of a triangle (2-simplex).

has many other characterizations which can be helpful for numerical computation of the ball radius.

PROPOSITION 3.2. *Let S be an m-simplex with circumscribed ball $B_C(c_C, r_C)$ and simplex ball $B_S(c_S, r_S)$ in $\mathbb{R}^n$. Let $B(c, r)$ be an m-dimensional ball in $\mathbb{R}^n$. Then, the following statements are equivalent:*

(i) $B = B_S$.

(ii) *B is an enfolding face ball of S with the smallest radius among all enfolding face balls of S.*

(iii) *B is a valid face ball of S with the largest radius among all valid face balls of S.*

(iv) *B is a ball with*

$$c = \operatorname{argmin}\{\max_{y \in S} \|x - y\|_2 \mid x \in S\} \qquad \text{and} \qquad r = \max_{y \in S} \|c - y\|_2.$$

(v) *B is a ball with*

$$c = \operatorname{argmin}\{\|x - c_C\|_2 \mid x \in S\} \qquad \text{and} \qquad r = \sqrt{r_C^2 - \|c - c_C\|_2^2}.$$

Proof. See [15, p. 36] for a verification. ∎

4. ERROR ESTIMATES

Now, we return to the main problem of finding estimates for the interpolation error E_∞. In a first step, we consider only real-valued functions. The following lemma describes the difference between a function and its linear interpolant on a simplex.

LEMMA 4.1. *Let S be an n-simplex with vertices $v_0, ..., v_n$ and circumscribed ball $B_C(c_C, r_C)$ in $\mathbb{R}^n$. Let $f: S \to \mathbb{R}$ be a real–valued function and $s = Lf$ the linear polynomial that interpolates f at the vertices of S.*

(i) *Using barycentric coordinates, for $x \in S$*

$$f(x) - s(x) = \sum_{i=0}^{n} \lambda_i(x)(f(x) - f(v_i)).$$

(ii) *If $f \in C^2(S)$, then for each $x \in S$, there exists $\xi \in S$ and $w \in \mathbb{R}^n \backslash \{0\}$ such that*

$$f(x) - s(x) = \frac{w^T f''(\xi)\, w}{2 w^T w}\, ((x - c_C)^T (x - c_C) - r_C^2).$$

Proof. (i) The representation of $f(x)-s(x)$ is an immediate consequence of (1) and (2).

(ii) For $x \in S$, $x \neq v_i$, we consider the auxiliary function $h: S \to \mathbb{R}$ with

$$h(z) := (f(z)-s(z)) - \frac{q(z)}{q(x)}(f(x)-s(x)),$$
$$q(z) := (z-c_C)^T (z-c_C) - r_C^2.$$

The function h vanishes at the v_i and at x, hence

$$h(x) = 0 = \sum_{i=0}^{n} \lambda_i(x)\, h(v_i),$$

showing that h fails to be strictly concave or convex on S, hence h'' must fail to be definite somewhere on S. So, there is $\xi \in S$ and $w \in \mathbb{R}^n \backslash \{0\}$ with $w^T h''(\xi)\, w = 0$. From

$$0 = w^T h''(\xi)\, w = w^T f''(\xi)\, w - \frac{w^T 2\mathrm{I} w}{q(x)}(f(x)-s(x))$$

we can conclude the assertion. ∎

The second lemma deals with the maximization of two special functions needed later in Theorem 4.1.

LEMMA 4.2. *Let S be an n-simplex with vertices $v_0, ..., v_n$, circumscribed ball $B_C(c_C, r_C)$, and simplex ball $B_S(c_S, r_S)$ in $\mathbb{R}^n$. Let $g_1, g_2: S \to \mathbb{R}$ be two non-negative functions defined by*

$$g_1(x) := \sum_{i=0}^{n} \lambda_i(x)\, \|x - v_i\|_2 \qquad \text{and} \qquad g_2(x) := r_C^2 - \|x - c_C\|_2^2.$$

Then, the following is true:

$$\text{(i)} \quad \max_{x \in S} g_1(x) = r_S$$

$$\text{(ii)} \quad \max_{x \in S} g_2(x) = r_S^2$$

Proof. The center c_S lies in the relative interior of a possibly lower-dimensional face F spanned by the set of vertices V of S. The distance from each $v \in V$ to c_S is r_S. Otherwise, a smaller enfolding ball could be constructed. This would contradict the definition of a simplex ball as the smallest enfolding ball (see also Proposition 3.2(iii)).

(i) Using the Cauchy–Schwarz inequality, g_1 can be estimated according to

$$\begin{aligned} g_1(x) &= \sum_{i=0}^{n} \left(\sqrt{\lambda_i(x)}\right)\left(\sqrt{\lambda_i(x)}\, \|x - v_i\|_2\right) \\ &\leqslant \sqrt{\sum_{i=0}^{n} \lambda_i(x)\, \|x - v_i\|_2^2} \\ &= \sqrt{\sum_{i=0}^{n} \lambda_i(x)\, \|(x - c_C) - (v_i - c_C)\|_2^2} \\ &= \sqrt{\|x - c_C\|_2^2 - 2\, \|x - c_C\|_2^2 + r_C^2} \\ &= \sqrt{g_2(x)}. \end{aligned}$$

From part (ii) it follows that $g_1 \leqslant r_S$ on S. Since $\|c_S - v\|_2 = r_S$ for all $v \in V$, $g_1(c_S) = r_S$. An alternative proof can be established with an idea of P. Shvartsman (see [5]). That proof avoids the Cauchy–Schwarz inequality and uses concave functions instead.

(ii) The assertion follows immediately from Proposition 3.2(v). However, the result can also be obtained from some geometric facts concerning simplices and spheres (cf. [6]), as follows. If $c_S = c_C$, the proof is straightforward. If $c_S \neq c_C$, $c_C - c_S$ is perpendicular to F since $V \subseteq \partial B_C$ and $V \subseteq \partial B_S$. With Pythagoras we have

$$\|c_C - c_S\|_2^2 = \|c_C - v\|_2^2 - \|v - c_S\|_2^2 = r_C^2 - r_S^2$$

for all $v \in V$. Since

$$\|x - c_C\|_2^2 = \|x - c_S\|_2^2 + 2(x - c_S)^T (c_S - c_C) + \|c_S - c_C\|_2^2,$$

therefore

$$r_C^2 - \|x - c_C\|_2^2 = r_S^2 - \|x - c_S\|_2^2 - 2(x - c_S)^T (c_S - c_C)$$

for all $x \in S$. Taking into account that $(x - c_S)^T (c_S - c_C) \geqslant 0$ on S, we arrive at

$$g_2(x) \leqslant r_S^2 - \|x - c_S\|_2^2 \leqslant r_S^2$$

with equality for $x = c_S$. ∎

Now we are prepared to state a first theorem about error estimates which includes extremal functions to prove the sharpness of all inequalities.

THEOREM 4.1. *Let S be an n-simplex with simplex ball $B_S(c_S, r_S)$ in $\mathbb{R}^n$. Let $f\colon S \to \mathbb{R}$ be a real-valued function and $s = Lf$ the linear polynomial that interpolates f at the vertices of S. Let $\omega(f, \cdot)$ denote the modulus of continuity of f. Then, the estimates*

(i) $E_\infty \leqslant K_1 \omega(f, 2r_S)$

(ii) $E_\infty \leqslant K_0 \, \|f\|_\infty$ *if* $f \in C^0(S)$

(iii) $E_\infty \leqslant K_1 r_S L$ *if f is Lipschitz continuous with constant L*

(iv) $E_\infty \leqslant K_1 r_S \, \|f'\|_{2,\infty}$ *if* $f \in C^1(S)$

(v) $E_\infty \leqslant K_2 r_S^2 \, \|f''\|_{2,\infty}$ *if* $f \in C^2(S)$

with the constants

$$K_0 = 2, \qquad K_1 = 1, \quad K_2 = \tfrac{1}{2}$$

hold, and the constants are best possible.

Proof. (i) Using Lemma 4.1(i), E_∞ can be estimated by

$$E_\infty \leqslant \max_{x \in S} \left(\sum_{i=0}^{n} \lambda_i(x) \, |f(x) - f(v_i)| \right) \leqslant \omega(f, 2r_S).$$

(ii) As an immediate consequence of (i), we obtain

$$E_\infty \leqslant \omega(f, 2r_S) \leqslant 2 \, \|f\|_\infty.$$

(iii) With Lemma 4.1(i) and Lemma 4.2(i), it is easy to see that

$$E_\infty \leqslant \max_{x \in S} \left(L \sum_{i=0}^{n} \lambda_i(x) \, \|x - v_i\|_2 \right) \leqslant L r_S.$$

(iv) Applying the mean value theorem, Lemma 4.1(i), and Lemma 4.2(i), we obtain

$$E_\infty \leqslant \max_{x, \xi_i \in S} \left(\sum_{i=0}^{n} \lambda_i(x) \, \|f'(\xi_i)\|_2 \, \|x - v_i\|_2 \right) \leqslant \|f'\|_{2,\infty} \, r_S.$$

(v) Let $B_C(c_C, r_C)$ be the circumscribed ball of S. With Lemma 4.1(ii) and Lemma 4.2(ii), we have

$$\begin{aligned} E_\infty &\leqslant \max_{\substack{x, \xi \in S \\ w \in \mathbb{R}^n \setminus \{0\}}} \left| \frac{w^T f''(\xi) \, w}{2 w^T w} \left((x - c_C)^T (x - c_C) - r_C^2 \right) \right| \\ &\leqslant \frac{1}{2} \max_{x \in S} \left(r_C^2 - \|x - c_C\|_2^2 \right) \|f''\|_{2,\infty} \leqslant \frac{1}{2} r_S^2 \, \|f''\|_{2,\infty}. \end{aligned}$$

In parts (i) and (ii), the distance function

$$f(x) := 2\,\|x - c_S\|_2 - r_S$$

is an example proving the sharpness of the constants. In fact, we have $E_\infty = 2r_S$, $\omega(f, 2r_S) = 2r_S$, and $\|f\|_\infty = r_S$. The power function

$$f(x) := \|x - c_S\|_2^\alpha, \qquad f'(x) = \alpha\,\|x - c_S\|_2^{\alpha-2}\,(x - c_S)^T$$

shows optimality of the constants in parts (iii) and (iv). In the limit $\alpha \to 1+$, we obtain $E_\infty = r_S^\alpha \to r_S$ and $L = \|f'\|_{2,\infty} = \alpha r_S^{\alpha-1} \to 1$. In part (v), we consider the quadratic function

$$f(x) := (x - c_S)^T\,(x - c_S), \qquad f'(x) = 2(x - c_S)^T, \quad f''(x) = 2\mathrm{I}.$$

Here, $E_\infty = r_S^2$ and $\|f''\|_{2,\infty} = 2$. ∎

In a third lemma, we discuss the relation between the diameter of a simplex and its simplex ball radius.

LEMMA 4.3. *Let S be an m-simplex with diameter h and simplex ball radius r_S in $\mathbb{R}^n$. Then, the estimates*

$$K_{min} h \leqslant r_S \leqslant K_{max} h$$

with the constants

$$K_{min} = \frac{1}{2}, \qquad K_{max} = \sqrt{\frac{m}{2(m+1)}}$$

hold, and the constants are best possible.

Proof. The lower inequality is straightforward. Since the simplex ball is enfolding, its radius is at least $h/2$. The orthogonal m-simplex

$$S_{min} := \operatorname{conv}\left\{-\frac{h}{2}\,\mathrm{e}_1, \frac{h}{2}\,\mathrm{e}_1, \ldots, \frac{h}{2}\,\mathrm{e}_m\right\}$$

in $\mathbb{R}^n$ with e_i denoting the ith unit vector in $\mathbb{R}^n$ has diameter h. Its simplex ball is centered at $c_S = (0, \ldots, 0)^T$. Hence, $r_S = h/2$. The upper inequality is a special case of Jung's theorem (see [10, 2.10.41, p. 200]). The regular m-simplex

$$S_{max} := \operatorname{conv}\left\{\frac{h}{\sqrt{2}}\,\mathrm{e}_1, \ldots, \frac{h}{\sqrt{2}}\,\mathrm{e}_{m+1}\right\}$$

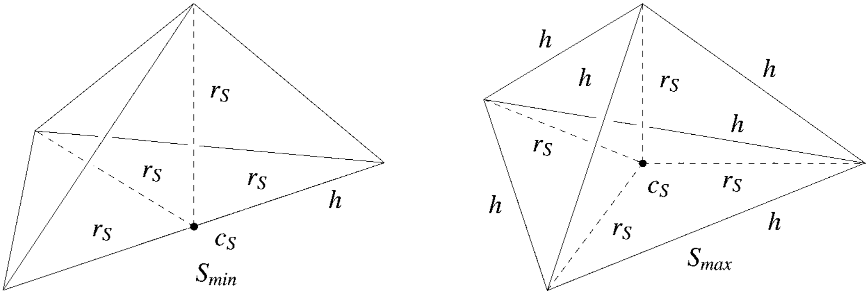

FIG. 2. Tetrahedron (3-simplex) with minimal and maximal simplex ball radius r_S with respect to the diameter h.

in $\mathbb{R}^{m+1}$ with e_i now denoting the ith unit vector in $\mathbb{R}^{m+1}$ has diameter h, and its simplex ball is centered at $c_S = ((h/\sqrt{2}\,(m+1)), ..., (h/\sqrt{2}\,(m+1))^T$ with $r_S = (\sqrt{m}/\sqrt{2(m+1)})\,h$. Moreover, any regular m-simplex in $\mathbb{R}^n$ with diameter h has this simplex ball radius. ∎

In Fig. 2, S_{min} and S_{max} are shown for the case $n = m = 3$. As a final preparation we introduce appropriate measures for the derivatives of a function $f = (f_1, ..., f_\ell)^T \in C^k(S)$ on a simplex S. The bounds of the kth derivative of f can be expressed by

$$M_k := \max_{1 \leqslant j \leqslant \ell} \left(\max_{x \in S} \left(\max_{\substack{w \in \mathbb{R}^n \\ \|w\|_2 = 1}} \left| \frac{\partial^k f_j}{\partial w^k}(x) \right| \right) \right), \qquad k = 0, 1, 2$$

in terms of derivatives in any directions, or by

$$\hat{M}_k := \max_{1 \leqslant j \leqslant \ell} \left(\max_{x \in S} \left(\max_{1 \leqslant v_1, ..., v_k \leqslant n} \left| \frac{\partial^k f_j}{\prod_{\mu=1}^{k} \partial x_{v_\mu}}(x) \right| \right) \right), \qquad k = 0, 1, 2$$

in terms of partial derivatives. The results of Theorem 4.1 can be generalized from real-valued to vector-valued functions, and from n-simplices to m-simplices with $m \leqslant n$ in $\mathbb{R}^n$. Moreover, the following theorem states the error in dependence of both the diameter and the simplex ball radius.

THEOREM 4.2. *Let S be an m-simplex with diameter h and simplex ball radius r_S in $\mathbb{R}^n$. Let $f = (f_1, ..., f_\ell)^T \colon S \to \mathbb{R}^\ell$ be a vector-valued function and $s = (s_1, ..., s_\ell)^T = Lf$ the linear polynomial that interpolates f at the vertices of S. Then, the estimates*

(i) $E_\infty \leqslant K_{0,1} M_0$ *if* $f \in C^0(S)$

(ii) $E_\infty \leqslant K_{1,1} r_S M_1 \leqslant K_{1,2} h M_1 \leqslant K_{1,3} h \hat{M}_1$ *if* $f \in C^1(S)$

(iii) $E_\infty \leqslant K_{2,1} r_S^2 M_2 \leqslant K_{2,2} h^2 M_2 \leqslant K_{2,3} h^2 \hat{M}_2$ *if* $f \in C^2(S)$

with the constants

$$K_{0,1} = 2\sqrt{\ell}$$

$$K_{1,1} = \sqrt{\ell}, \qquad K_{1,2} = \frac{1}{\sqrt{2}} \sqrt{\frac{m}{m+1}} \sqrt{\ell}, \quad K_{1,3} = \frac{1}{\sqrt{2}} \sqrt{\frac{m}{m+1}} \sqrt{n} \sqrt{\ell}$$

$$K_{2,1} = \frac{1}{2}\sqrt{\ell}, \qquad K_{2,2} = \frac{1}{4}\frac{m}{m+1}\sqrt{\ell}, \quad K_{2,3} = \frac{1}{4}\frac{m}{m+1} n \sqrt{\ell}$$

hold, and those constants preceding M_0, M_1, *and* M_2 *are best possible.*

Proof. Notice that Theorem 4.1 is also valid for arbitrary m-simplices with $0 \leqslant m \leqslant n$. For all three smoothness classes, we can permute norm operator and maximizing operator, and obtain

$$E_\infty = \max_{x \in S} \|f(x) - s(x)\|_2 \leqslant \left\| \begin{pmatrix} \max_{x \in S} |f_1(x) - s_1(x)| \\ \vdots \\ \max_{x \in S} |f_\ell(x) - s_\ell(x)| \end{pmatrix} \right\|_2$$
$$\leqslant \sqrt{\ell} \max_{1 \leqslant j \leqslant \ell} \left(\max_{x \in S} |f_j(x) - s_j(x)| \right). \tag{3}$$

Now, it is possible to apply Theorem 4.1 in each component of the error vector.

(i) If $f \in C^0(S)$, we conclude from (3)

$$E_\infty \leqslant \sqrt{\ell} \max_{1 \leqslant j \leqslant \ell} (2 \|f_j\|_\infty) = 2\sqrt{\ell} M_0.$$

(ii) If $f \in C^1(S)$, (3) yields

$$E_\infty \leqslant \sqrt{\ell} \max_{1 \leqslant j \leqslant \ell} (r_S \|f_j'\|_{2,\infty}) = \sqrt{\ell}\, r_S M_1.$$

The second and third inequality follow from Lemma 4.3 and

$$\|f_j'(x)\|_2 \leqslant \sqrt{n} \max_{1 \leqslant \nu \leqslant n} \left| \frac{\partial f_j}{\partial x_\nu}(x) \right|.$$

(iii) If $f \in C^2(S)$, using (3), the interpolation error is bounded by

$$E_\infty \leqslant \sqrt{\ell} \max_{1 \leqslant j \leqslant \ell} \left(\tfrac{1}{2} r_S^2 \, \|f_j''\|_{2,\infty}\right) = \tfrac{1}{2} \sqrt{\ell}\, r_S^2 M_2.$$

Let $\|\cdot\|_F$ denote the Frobenius norm of a matrix. Since the spectral norm is the smallest matrix norm, we have

$$\|f_j''(x)\|_2 \leqslant \|f_j''(x)\|_F \leqslant n \max_{1 \leqslant \nu_1, \nu_2 \leqslant n} \left| \frac{\partial^2 f_j}{\partial x_{\nu_1} \partial x_{\nu_2}}(x) \right|.$$

Once more, inserting Lemma 4.3 completes the estimations.

To see that the constants in terms involving M_0, M_1, and M_2 are best possible, consider vector-valued functions that consist of the example functions of Theorem 4.1 in each component. For these functions on regular simplices, all estimates are sharp. ∎

5. DISCUSSION AND PREVIOUS RESULTS

The circumscribed ball radius, the simplex ball radius, and the diameter measure the size of a simplex in different ways. All three quantities are independent of geometric simplex properties such as angles or edge length proportions. The radius of a circumscribed ball tends to infinity if any simplex angle tends to zero. In contrast, the radius of a simplex ball as well as the diameter stays bounded. Hence, the simplex ball radius and the diameter are better suited to describe the interpolation error. In general, the diameter is more accessible for numerical computation than a ball radius. Similar arguments apply with respect to the derivatives. Sharp inequalities can be obtained when admitting derivatives in any directions, whereas partial derivatives are the better choice for computational purposes. To be precise, Theorem 4.2 contains two slightly different types of sharpness. For all simplices there exist functions that turn the inequalities with $K_{k,1}$ into equalities. The sharpness of the inequalities with $K_{k,2}$ is in some sense weaker since there are functions with which equality can be achieved only on regular simplices.

Finally, we compare the results of Theorem 4.1 and Theorem 4.2 with previous results. All inequalities are well known in the univariate case. Some authors preferred alternative approaches that involve univariate as well as multivariate Taylor expansions of f around vertices and auxiliary points. Prenter [14, p. 142], e.g., followed such a method for $n = m = 2$, $\ell = 1$, and $f \in C^2$, and obtained a constant of 4 for $\hat{M}_2$. Hämmerlin [11, p. 243] reduced the value to $\frac{3}{2} + \sqrt{3}$. In comparison with these early results, the constant $K_{2,3} = \frac{1}{3}$ is a significant improvement. Subbotin proved inequalities involving $K_{2,2}$ in the bivariate case (see [16, Proposition 1]) and in the general case (see [17, Theorem 1]) using an inductive argument. In [5], the derivation of $K_{1,1}$ is based on an argument of P. Shvartsman (cf. Lemma 4.2). For the bivariate case and $\ell = 1$, Handscomb [12, p. 14] derived $K_{2,1}$ for acute-angled triangles, and the constant $\frac{1}{8}$ otherwise. The latter constant is valid only for obtuse-angled triangles, whereas $K_{2,2} = \frac{1}{6}$ is the smallest constant for triangles of general shape. In a recent paper, Waldron used divided difference functionals to obtain $K_{2,1}$ (cf. [18, Theorem 3.1.]), but writes the term r_S^2 in the more complicated way $r_C^2 - \mathrm{dist}^2(c_C, S)$. L_p bounds are also considered in that paper. Most constants of Theorem 4.2 and a discussion of simplex balls can be found in [15, p. 31].

6. CONCLUSION

Roughly speaking, the interpolation error depends linearly or quadratically on the simplex size if $f \in C^1$ or $f \in C^2$, respectively. The estimates presented in this paper and consequently the actual linear interpolation error are bounded on simplices of any shape. This is contrary to the popular fallacy within the FEM community which states that triangles or tetrahedrons with small angles are bad. The dimension ℓ of the image space of f as well as the dimension n of the embedding space and the dimension m of the simplex have an influence on the estimates. Future work can be done on the question of how these results can be generalized to higher-order interpolation.

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